C 20207 **(Pages : 3)** **Name..................................**

Reg. No...............................

SIXTH SEMESTER (CUCBCSS–UG) DEGREE EXAMINATION, MARCH 2022

Mathematics

MAT 6B 09—REAL ANALYSIS

(2014 to 2018 Admissions)

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Define uniform continuity of a function.
2. State Weierstrass approximation theorem.
3. Find $\|P\|$ if P = {0, 2, 3, 4} in a partition of [0, 4].
4. Give an example for a function which is not Riemann integrable.
5. Define step function.
6. State Lebesgue integrability criterion.
7. Define uniform convergence of a series of functions.
8. $\lim\limits_{n\to\infty}\dfrac{x^2+nx}{n}$.
9. Write an example for an absolutely convergent improper integral.
10. Cauchy principal value of $\int\limits_{-\infty}^{\infty} x\,dx =$.
11. Define Beta function.
12. Fill in the blanks : $\Gamma(3) =$ ————.

(12 × 1 = 12 marks)

Part B

Answer any **ten** *questions.*
Each question carries 4 *marks.*

13. Let I = [*a, b*] be a closed boudned interval and $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying $\inf f(I) \le k \le \sup f(I)$ then prove that there exists a number $c \in I$ such that $f(c) = k$.
14. State and prove Preservation of Intervals theorem.

Turn over

15. Show by an example that every uniformly continuous function need not be a Lipschitz function.
16. If $\phi:[a,b]\to\mathbb{R}$ is a step function, prove that $\phi\in\mathfrak{R}[a,b]$.
17. Suppose that $f,g\in\mathfrak{R}[a,b]$. Prove that $fg\in\mathfrak{R}[a,b]$.
18. State the substitution theorem of Riemann integration. Use it to evaluate $\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt$.
19. Let (f_n) be a sequence of bounded functions of $A\subseteq\mathbb{R}$. Suppose that $\|f_n-f\|_A\to 0$. Then prove that (f_n) converges uniformy on A to f.
20. If f_n is continuous of $D\subseteq\mathbb{R}$ to $\mathbb{R}$ for each $n\in\mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then prove that f is continuous on D.
21. State and prove Weierstrass M-Test for a series of functions.
22. Discuss the uniform convergence of $\sum_{n=1}^{\infty}\frac{1}{x^2+n^2}$.
23. Test the convergence of $\int_0^{\infty}\frac{1}{x^2}\,dx$.
24. Show that $\Gamma(n+1)=n!$ when n is a positive integer.
25. Show that $\beta(m,n)=\beta(n,m)$.
26. Evaluate $\int_0^1 x^m\left(\log\frac{1}{x}\right)^n dx$.

(10 × 4 = 40 marks)

Part C

Answer any **six** *questions.*
Each question carries 7 marks.

27. Let $I=[a,b]$ be a closed bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Then prove that f is bounded on I.
28. State and prove Uniform Continuity Theorem.
29. State and prove Continuous Extension Theorem.
30. If $f:[a,b]\to\mathbb{R}$ is monotone on $[a,b]$ then prove that $f\in\mathfrak{R}[a,b]$.
31. Discuss the convergence of the sequence $(f_n(x))$ where $f_n(x)=\frac{x^n}{x^n+1}$, $x\in[0,2]$.
32. State and prove Taylor's Theorem with the Reminder.

33. Let $f \in \mathfrak{R}[a,b]$ and let f be continuous at a point $c \in [a,b]$. Prove that the indefinite integral $F(z) = \int_a^z f$ foe $z \in [a,b]$ is differentiable at c and $F'(c) = f(c)$.

34. Show that $\Gamma\left(\frac{P}{2}\right)\Gamma\left(\frac{P+1}{2}\right) = \frac{\sqrt{\pi}}{2^{P-1}}\Gamma(p)$.

35. Evaluate the integral $\int_0^1 x^2\left(1-\sqrt{x}\right)dx$.

(6 × 7 = 42 marks)

Part D

Answer any **two** *questions.*
Each question carries 13 *marks.*

36. (a) State and prove Maximum Minimum Theorem.

(b) Test the uniform continuity of $f(x) = \sqrt{x}$ on [0, 2].

37. (a) Let $f:[a,b] \to \mathbb{R}$ and $c \in (a,b)$. Prove that $f \in \mathfrak{R}[a,b]$ if and only if its restriction to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case show that $\int_a^b f = \int_a^c f + \int_c^b f$.

(b) If $f \in \mathfrak{R}[a,b]$ and if $[c,d] \subseteq [a,b]$ then prove that the restriction of f to $[c, d]$ is in $\mathfrak{R}[c, d]$.

38. (a) Prove that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \forall m, n > 0$.

(b) Find the value of $\Gamma\left(\frac{1}{2}\right)$.

(2 × 13 = 26 marks)